LBL-47721
Mc-Gill-preprint-01-09
GSI-Preprint/2001-10

Fluctuations of rare particles as a measure of chemical equilibration

S. Jeon[a†*], V. Koch[a], K. Redlich[a,c], and X.-N. Wang[a]
[a] *Nuclear Science Division, Lawrence Berkeley National Laboratory, 1 Cyclotron Road, Berkeley, CA 94720*
[c] *GSI, D-64291 Darmstadt, Germany and Institute of Theoretical Physics, University of Wrocław, PL-50204 Wrocław, Poland*
†*Department of Physics, McGill University, 3600 University Street, Montreal, QC H3A-2T8, Canada*
∗*RIKEN-BNL Research Center, Brookhaven National Laboratory, Upton, NY 11973*
(August 17, 2001)

We calculate the time evolution of fluctuations for rare particles such as e.g. kaons in 1 AGeV or charmonium in 200 AGeV heavy ion collisions. We find that these fluctuations are a very sensitive probe of the degree of chemical equilibration reached in these collisions. Furthermore, measuring the second factorial moment the size of the initial population can be determined.

I. INTRODUCTION

Statistical models have long been used as a tool to describe particle production in heavy ion and in high energy particle collisions [1–3]. Recent analysis have shown that these models can indeed give a satisfactory description of the multiplicities of most hadrons measured in A-A collisions for bombarding energies ranging from 1 AGeV (SIS) to 160 AGeV (SPS) [4,5]. Especially at the low energies (SIS) and for the rare particles such as kaons, the success of the statistical description has been a puzzle and could not be understood within the state-of-the-art transport models [6]. For instance a recent analysis for chemical equilibration within a transport model [6] gives chemical equilibration time of the order of 300 fm/c for a situation relevant to 1 AGeV heavy ion collisions, considerably larger than the typical duration of such reaction, which is of the order of 30-50 fm/c. On the other hand the success of the statistical model cannot be disputed. Not only are the particle ratios reproduced for central collision over a wide range of bombarding energy, but also the centrality dependence at low energies is consistent with the predictions of the statistical models. In particular the almost *quadratic* dependence of the kaon multiplicity on the number of participating nucleon at SIS follows directly from the statistical model, once strangeness conservation is taken into account exactly [7]. Transport models on the other hand fail to reproduce the observed centrality dependence and particularly energy dependence of the K/π-ratio [8].

In a recent paper [9] some of us have shown, that the chemical equilibration time is considerably shortened if the strangeness conservation is taken into account explicitly. However, using the new rate equations derived in this work combined with the cross sections as given in [14] one still arrives at equilibration time substantially exceeding the lifetime of the system.

Does this mean that the success of the statistical model is a pure coincidence? This is very unlikely as statistical model naturally explains most of the basic features of experimental data in a very broad energy range from SIS up to SPS. It is conceivable that there are additional processes at work, like e.g. many particle collisions [15] or in medium modifications of hadron properties [16], which are not yet taken into account in transport models.

Thus, a direct experimental determination of the rate of equilibration in heavy ion collisions is called for as it would possibly provide evidence for new physics. In this paper we will demonstrate that the fluctuations of rare particles is a very sensitive probe of the degree of equilibration reached in these collisions. Such a measurement, though certainly difficult, could for the first time provide a direct experimental evidence for chemical equilibration in heavy ion reactions.

This paper is organized as follows. In the following section we set up the formalism. Then we present the results for the time dependence of the second factorial moment for several initial conditions. Before we turn to observational issues we also will discuss the case when no constraints from any $U(1)$ charge conservation are present.

II. FORMALISM

In Ref. [9] the rate equation for particles which are subject to an explicit $U(1)$ "charge" conservation has been derived. Considering a binary process $a_1 a_2 \to b_1 b_2$ with $a \neq b$ one arrives at the following master equation for the probabilities P_n to find n particles

$$\frac{dP_n}{d\tau} = \epsilon \left[P_{n-1} - P_n \right]$$

$$-\left[n^2 P_n - (n+1)^2 P_{n+1}\right], \tag{1}$$

where $n = 0, 1, 2, 3, \cdots$. Here

$$\epsilon \equiv G \langle N_{a_1} \rangle \langle N_{a_2} \rangle / L, \tag{2}$$

and the dimensionless time variable τ is defined as

$$\tau = t \frac{L}{V} \tag{3}$$

so that τ is measured in units of the relaxation time $\tau_0^C = V/L$ [9]. The momentum-averaged cross sections for the gain process $a_1 a_2 \to b_1 b_2$ and the loss process $b_1 b_2 \to a_1 a_2$ are defined as $G \equiv \langle \sigma_G v \rangle$ and $L \equiv \langle \sigma_L v \rangle$, respectively. The ratio of these momentum averaged cross sections is related to the ratio of equilibrium particle densities involved

$$\frac{G}{L} = \frac{d_{b_1} \alpha_{b_1}^2 K_2(\alpha_{b_1}) d_{b_2} \alpha_{b_2}^2 K_2(\alpha_{b_2})}{d_{a_1} \alpha_{a_1}^2 K_2(\alpha_{a_1}) d_{a_2} \alpha_{a_2}^2 K_2(\alpha_{a_2})}, \tag{4}$$

where d_k's denote the degeneracy factors, and $\alpha_k \equiv m_k/T$.

Eq. (1) has no obvious solution and needs to be solved numerically. The asymptotic (equilibrium) probability distribution, on the other hand, has been derived in [9]

$$P_{n,\text{eq.}} = \frac{\epsilon^n}{I_0(2\sqrt{\epsilon})\ (n!)^2}. \tag{5}$$

leading to

$$\langle N \rangle_{\text{eq.}} = \sqrt{\epsilon} \frac{I_1(2\sqrt{\epsilon})}{I_0(2\sqrt{\epsilon})} = \epsilon - \frac{\epsilon^2}{2} + \frac{\epsilon^3}{3} + \ldots \tag{6}$$

$$\langle N^2 \rangle_{\text{eq.}} = \epsilon. \tag{7}$$

The above general rate equation is valid for arbitrary values of $\langle N \rangle$ for particle production constrained by $U(1)$ charge conservation. It reduces to the grand canonical results for large $\langle N \rangle$ and to the canonical results for small $\langle N \rangle$. It provides a generalization of the standard rate equation beyond the grand canonical limit. It was shown [9] that for rare particle production the equilibrium abundance is much smaller and the relaxation time is much shorter than expected from the standard rate equation. In this paper we will discuss further consequences of the generalized rate equation and in particular study the time evolution of the multiplicity fluctuations. We want to demonstrate that the combined information on both $\langle N \rangle$ and $\langle N^2 \rangle$ can help to determine the degree of chemical equilibration.

III. RESULTS

In this work we will be mostly concerned with the fluctuations of the particle number in the case of rare particle, i.e. $\langle N \rangle_{\text{eq.}} \ll 1$ or, equivalently, $\epsilon \ll 1$. In particular we will investigate the behavior of the second factorial moment F_2

$$F_2 \equiv \frac{\langle N(N-1) \rangle}{\langle N \rangle^2}. \tag{8}$$

From Eqs. (6) and (7) in the limit of small ϵ the equilibrium value for F_2 is given by

$$F_2 = \frac{1}{2} + \frac{\epsilon}{6} + \ldots = \frac{1}{2} + \frac{\langle N \rangle_{\text{eq.}}}{6} + \ldots \tag{9}$$

In order for the second factorial moment to be a sensitive probe of the degree of equilibrium achieved, one needs to investigate its initial value. Here we consider two distinct cases:

1. The initial particle number is considerably smaller than the equilibrium value. This is relevant, for example, for kaon production in 1 AGeV heavy ion collisions.
2. The initial particle number is considerably larger than the equilibrium value. This might be relevant for charm production in 200 AGeV heavy ion collisions [17]

In the first case, where the initial particle number is small, let us consider two scenarios. On one hand, let us assume that initially the probabilities P_n are distributed according to a Poisson distribution:

$$P_n(\tau = 0) = \frac{N_0^n}{n!} e^{-N_0}, \tag{10}$$

where N_0 is the initial average number of particles. In this case, the factorial moment obviously starts out at

$$F_2(\tau = 0) = 1 \tag{11}$$

and decreases by a factor of two until equilibrium is reached. On the other hand one may assume that initially there is at most one particle in a given event. In this case the initial conditions are

$$\begin{aligned} &P_0(\tau = 0) = 1 - N_0, \\ &P_1(\tau = 0) = N_0, \\ &P_n(\tau = 0) = 0 \quad n > 1, \end{aligned} \tag{12}$$

which we will refer to as 'binomial' initial conditions. As shown in Appendix A, F_2 starts out at $F_2 = 0$, but almost immediately reaches a maximum after a time of the order of

$$\tau_{\max} \simeq \frac{N_0}{N_{\text{eq.}}} \tag{13}$$

and for $N_0/N_{\text{eq.}} \ll 1$, $F_2^{\max} \simeq 1$.

Therefore F_2 approaches equilibrium from above and a measurement of $F_2 > 1/2$ will indicate the degree of equilibrium that has been reached in a heavy ion collision. The detailed time and height, where F_2 reaches a maximum depend, of course, on the input parameters N_0 and ϵ. The dependence of $F_2^{\max}$ on the ratio $N_0/N_{\text{eq.}}$ for $\epsilon = 0.1$ is shown in Fig. 1 as the full line. The dashed line in Fig. 1 is obtained by assuming $\tau_{\max} = 3N_0/N_{\text{eq.}}$ showing that indeed the time scale for reaching the maximum is given by $N_0/N_{\text{eq.}}$.

We further see that for small N_0 the factorial moment essentially immediately reaches a value close to $F_2 = 1$, giving a factor of two sensitivity on the degree of non-equilibrium established in the collisions. Obviously, in the case where $N_0 \simeq \epsilon = N_{\text{eq.}}$ this sensitivity is lost, as the equilibrium value is very close to the initial value.

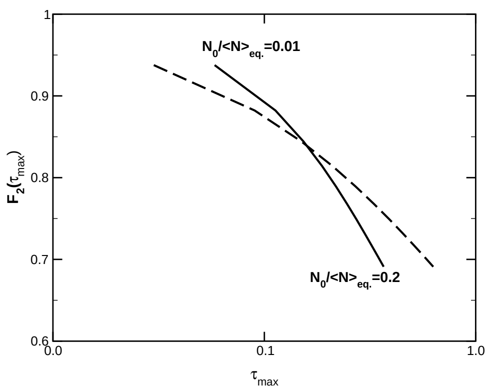

FIG. 1. Location and value of the maximum of $F_2(\tau)$ for a range of values of $N_0/N_{\text{eq.}}$ as indicated in the figure. Here $\epsilon = 0.1$ has been used. The dashed line assumes that $\tau_{\max} = 3N_0/N_{\text{eq.}}$

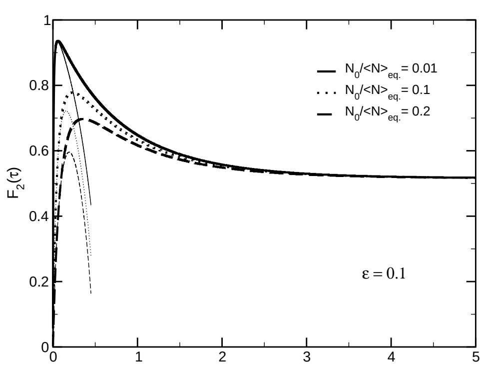

FIG. 2. Time evolution of the factorial moment F_2 for several initial particle numbers N_0 (thick lines). The thin lines show the result of the approximate formula (A2). Here $\epsilon = 0.1$ has been used.

Assuming binomial initial conditions in Fig. 2 we show the full time evolution for several initial particle numbers. For small times the approximate solution (A2) is also shown. Clearly, the equilibrium value of F_2 is reached from above, but the effect becomes small as the initial particle number becomes comparable with the equilibrium value.

In Fig. 3 we show the time evolution of F_2 for different choices of ϵ or, equivalently, for different equilibrium particle numbers. Obviously, the larger the equilibrium particle number is the smaller is the effect.

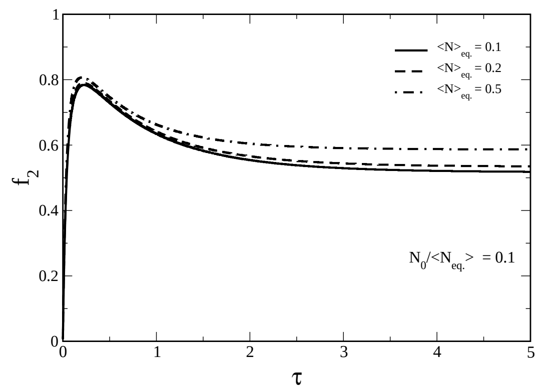

FIG. 3. Time evolution of the factorial moment F_2 for several equilibrium numbers or equivalently values of ϵ. Here binomial initial conditions with $N_0/N_{\text{eq.}} = 0.1$ have been chosen.

For the second case when the initial population is much bigger than the equilibrium population, then the annihilation process must dominate during the early stage of evolution for a short period of time. In this case, one can look for a perturbative solution around $\epsilon = 0$. Since there can be many different initial conditions with a large initial population, it is better here to use the generating function [9]

$$g(\tau, x) = \sum_{n=0}^{\infty} x^n P_n(\tau), \tag{14}$$

and the equation it satisfies

$$\frac{\partial g}{\partial \tau} = (1 - x)\left(xg'' + g' - \epsilon g\right) \ . \tag{15}$$

where the prime indicates a derivative with respect to x. The averages needed to calculate the first two factorial moments are given by

$$\langle N \rangle = g'(\tau, 1) \quad \text{and} \quad \langle N(N-1) \rangle = g''(\tau, 1). \tag{16}$$

Details of the perturbative procedure can be found in Appendix B. To understand the qualitative picture, first consider the initial time $\tau = 0$. Since we assume $\langle N \rangle \gg 1$ initially, we must have $\langle N(N-1) \rangle = \mathcal{O}(\langle N \rangle^2)$ and hence,

$F_2 = \mathcal{O}(1)$ at $\tau = 0$. For $\tau \gtrsim 1$, the first order perturbative solutions are

$$\langle N\rangle = \langle N\rangle_{\rm eq.} + |a_1|\, e^{-\tau} + \mathcal{O}(e^{-4\tau}), \tag{17}$$

$$\langle N(N-1)\rangle = \langle N(N-1)\rangle_{\rm eq.} + \frac{2\,\epsilon}{5}\,|a_1|e^{-\tau} + \mathcal{O}(e^{-4\tau}), \tag{18}$$

where a_1 is a $\mathcal{O}(1)$ constant determined by the initial condition. From the above equations and using Eqs. (6) and (7) the second factorial moment is then given by

$$F_2(\tau) \simeq \frac{1}{2}\frac{\epsilon^2 + (4/5)\epsilon\,|a_1|\,e^{-\tau}}{(\epsilon + |a_1|\,e^{-\tau})^2}. \tag{19}$$

For times $1 \lesssim \tau \lesssim -\ln\epsilon$, the exponential terms in Eq. (19) dominate. As a result, $F_2(\tau) = \mathcal{O}(\epsilon)$ within the interval $1 \lesssim \tau \lesssim -\ln\epsilon$. Note that for arbitrarily small ϵ, this interval can be arbitrarily long. Also, since $\mathcal{O}(\epsilon) \ll F_2(0)$ and $\mathcal{O}(\epsilon) \ll F_2^{\rm eq.}$, F_2 must reach a minimum somewhere inside that interval.

For illustration, let us choose a Poissonian initial distribution with $N_0 = 5$ and also set $\epsilon = 0.1$. The numerical solutions and Eq. (19) as well as the second order perturbative solution are displayed in Fig. 4. The numerical solution clearly shows the rapid initial decrease and the subsequent slower rise to the equilibrium value. To illustrate the duration of the small F_2 interval, we also show the full numerical result with $\epsilon = 0.001$. The longer period of time where F_2 stays to be $\mathcal{O}(\epsilon)$ is clearly visible.

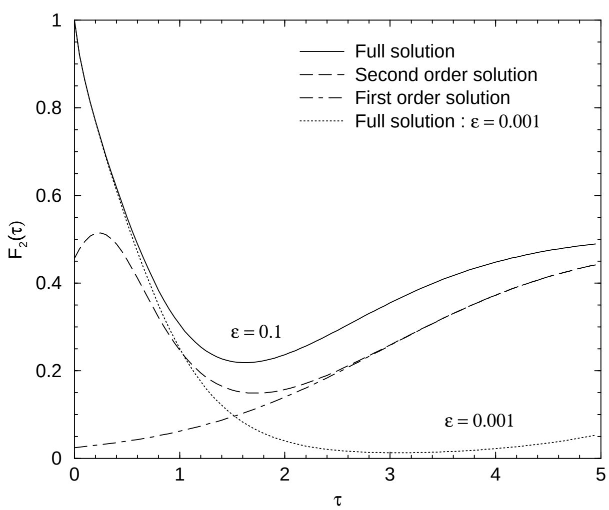

FIG. 4. The second factorial moment as a function of time. Initial distribution is a Poisson distribution with $N_0 = 5$. $\epsilon = 0.1$. The solid line represents the numerical solution. The dashed line is the result of the second order perturbative calculation and the dot-dashed line is the result of the first order perturbative calculation. Also shown is the full numerical solution with the same initial condition and $\epsilon = 0.001$.

To sum up, F_2 as a function of time must:

(i) Start from $\mathcal{O}(1)$.

(ii) Reach the minimum value of $\mathcal{O}(\epsilon)$ at $\tau \sim 1$ which is much smaller than $F_2^{\rm eq.} = 1/2 + \mathcal{O}(\epsilon)$.

(iii) Stay $\mathcal{O}(\epsilon)$ until $\tau \sim -\ln\epsilon$.

(iv) Approach the equilibrium value from below after $\tau \sim -\ln\epsilon$. This is in contrast to the first case where we considered $N_0 \ll \langle N\rangle_{\rm eq.}$.

Hence, if experimental value of F_2 is smaller than $1/2$, then it is a strong indication that the equilibrium is not reached and furthermore it also indicates that the initial population was much larger than the equilibrium one.

IV. ABSENCE OF $U(1)$-CHARGE CONSERVATION

It is interesting to also study particle production without the constraint of $U(1)$ charge conservation. Obviously the time evolution equation for the multiplicity distribution should be different from what we have described so far for strongly correlated processes. Let us consider a general process $a + b \leftrightarrow c + d$ without any constraint of charge conservation. We denote P_n as the multiplicity distribution for particle c. It then satisfies the following evolution equation,

$$\frac{dP_n}{d\tilde\tau} = \tilde\epsilon[P_{n-1} - P_n] - [nP_n - (n+1)P_{n+1}], \tag{20}$$

where $\tilde\tau = \tau/\langle N_d\rangle = tL/V/\langle N_d\rangle$ is a scaled time and $\tilde\epsilon = \epsilon/\langle N_d\rangle = G\langle N_a\rangle\langle N_b\rangle/L/\langle N_d\rangle$.

The generating function $g(\tilde\tau, x)$ for P_n

$$g(\tilde\tau, x) = \sum_{n=0}^{\infty} x^n P_n(\tilde\tau) \tag{21}$$

satisfies the following partial differential equation,

$$\frac{\partial g}{\partial\tilde\tau} = (1-x)[g' - \tilde\epsilon g]. \tag{22}$$

It is interesting to note that the above equation is very similar to Eq. (15) for a constrained system except that it does not contain the second derivative on the right-hand side. Therefore, for certain period during which one can neglect the second derivative of the generating functions, the evolution of the multiplicity distribution in a canonical system should be similar to that of a grand canonical.

The general solution to the Eq. (22) can be found

$$g(\tilde\tau, x) = g_0((1-x)e^{-\tilde\tau})e^{\tilde\epsilon(1-x)(e^{-\tilde\tau}-1)} \tag{23}$$

with the initial condition $g(0, x) \equiv g_0(1-x)$. The normalization condition $g(\tilde\tau, x = 1) = \sum P_n = 1$ also implies $g_0(0) = 1$. One can readily find the equilibrium solution in the limit $\tilde\tau = \infty$,

$$g_{\text{eq.}}(x) = e^{-\tilde{\epsilon}(1-x)} \tag{24}$$

with the corresponding equilibrium multiplicity distribution

$$P_{n,\text{eq.}} = \frac{\tilde{\epsilon}^n}{n!} e^{-\tilde{\epsilon}}, \tag{25}$$

which is a Poisson distribution with averaged multiplicity $\langle N \rangle_{\text{eq.}} = \tilde{\epsilon}$. One can also easily calculate the first and second factorial moments of the multiplicity distribution,

$$\langle N \rangle = \epsilon + (\langle N \rangle_0 - \epsilon) e^{-\tilde{\tau}} \tag{26}$$

$$\begin{aligned} \langle N(N-1) \rangle &= \langle N(N-1) \rangle_0 e^{-2\tilde{\tau}} \\ &- 2\epsilon \langle N \rangle_0 e^{-\tilde{\tau}} (e^{-\tilde{\tau}} - 1) + \epsilon^2 (e^{-\tilde{\tau}} - 1)^2. \end{aligned} \tag{27}$$

They both approach to their equilibrium values exponentially.

One interesting case of a special initial condition is $g_0 = 1$ when the initial multiplicity $\langle N \rangle_0$ is zero. Comparing Eqs.(23) and (24), one finds that the multiplicity distribution in this case remains a Poissonian, and consequently the factorial moment $F_2 = 1$ at all times. For the binomial initial conditions (Eq.(12)),

$$g(\tilde{\tau} = 0, x) = 1 - N_0(1 - x), \tag{28}$$

the factorial moment F_2 starts out from $F_2(\tilde{\tau} = 0) = 0$, but approaches the equilibrium value via

$$F_2 = 1 - [N_0 / (\tilde{\epsilon}(e^{\tilde{\tau}} - 1) + N_0)]^2 \tag{29}$$

at a time scale of

$$\tilde{\tau} \simeq \ln(1 + \sqrt{2} N_0 / \tilde{\epsilon}) \simeq \sqrt{2} N_0 / \tilde{\epsilon}. \tag{30}$$

Comparing the results presented here with the previous section it is clear that an additional constraints imposed by the U(1) charge conservation laws are implying a crucial modification of not only the equilibrium probability distributions of particle number but also their fluctuations and time evolution towards the equilibrium limit.

V. TOWARDS EXPERIMENTAL OBSERVABLES

So far we have discussed the production of a single species of particles with conserved quantum numbers, such as e.g. K^+-K^--pairs. In reality, however, one has to deal with more than one species. For example at 1 AGeV heavy-ion collisions the relevant strange degrees of freedom are K^+ and K^0 which carry positive strangeness and the Λ and Σ hyperons carrying negative strangeness. The anti-kaons are not relevant in this case as the ratio of K^-/K^+ is very small ($\simeq 2\%$ at 1.5 AGeV [10]). The results shown in the previous section thus apply to the combined multiplicities for K^+ and K^0, i.e.

$$N = N_{K^+} + N_{K^0}. \tag{31}$$

Very often, experiments can either measure K^+ or K^0 but not both species at the same time. The above master equation has been extended for more than one particle species carrying the conserved quantum number in reference [11]. The equation for the combined probability $P_{i,j}$ to find i K^+ and j K^0 mesons is given by

$$\begin{aligned} \frac{dP_{i,j}}{d\tau} &= \epsilon_1 (P_{i-1,j} - P_{i,j}) + \epsilon_2 \frac{L_2}{L_1} (P_{i,j-1} - P_{i,j}) \\ &\quad - (i(i+j) P_{i,j} - (i+1)(i+j+1) P_{i+1,j}) \\ &- \frac{L_2}{L_1} (j(i+j) P_{i,j} - (j+1)(i+j+1) P_{i,j+1}), \end{aligned} \tag{32}$$

where $\tau = t \frac{L_1}{V}$ and $\epsilon_{1,2} \equiv G_{1,2} \langle N_{a_1} \rangle \langle N_{a_2} \rangle / L_{1,2}$. The equilibrium solution is given by [11]

$$P_{i,j}^{\text{eq.}} = \frac{\epsilon_{tot}^{i+j}}{I_0(2\sqrt{\epsilon_{tot}})((i+j)!)^2} \frac{(i+j)! \epsilon_1^i \epsilon_2^j}{\epsilon_{tot}^{i+j} i! j!} \tag{33}$$

with $\epsilon_{tot} \equiv \epsilon_1 + \epsilon_2$. Note that the equilibrium probability distribution is the product of the distribution of pairs according to Eq. (5) and a binomial distribution, which determines the relative weight of the individual particles, in our case the K^+ and K^0. For the equilibrium configuration the relevant expectation values are then easily computed

$$\langle N_1 \rangle_{\text{eq.}} = f_1 \langle N \rangle_{\epsilon_{tot}}, \tag{34}$$

$$\langle N_1^2 \rangle_{\text{eq.}} = f_1^2 \langle N^2 \rangle_{\epsilon_{tot}} + f_1(1 - f_1) \langle N \rangle_{\epsilon_{tot}}, \tag{35}$$

$$F_2^{equil}(K^+) = \frac{1}{2} + \frac{\epsilon_{tot}}{6} + \ldots \tag{36}$$

where $f_1 = \epsilon_1 / \epsilon_{tot}$. The average $\langle \rangle_{\epsilon_{tot}}$ denotes the averages given in Eqs. (6) and (7) with $\epsilon \to \epsilon_{tot}$ For small $\epsilon_{1,2}$ the effect of the second species only appears at next to leading order in F_2.

The master equation governing the particle of interest, say the K^+ is obtained by summing equation (32) over the index of the particle which is not observed.

$$\begin{aligned} \frac{dP_i}{d\tau} &= \epsilon_1 (P_{i-1j} - P_i) \\ &\quad - (i^2 P_i - (i+1)^2 P_{i+1}) \\ &\quad - (i \sum_j j P_{i,j} - (i+1) \sum_j j P_{i+1,j}) \end{aligned} \tag{37}$$

with

$$P_i \equiv \sum_j P_{i,j}. \tag{38}$$

Comparing with the original equation (1), the presence of the other species, the K^0 in our case, leads to the last two terms of equation (37). However, in the situation of interest here, where $N_{K^0} \ll 1$ these terms can be neglected. Thus we recover the original equation governing the time evolution of the K^+. This can also be seen in

Fig. (5) where we compare the evolution of F_2 based on Eq. (1) with that based on Eq. (37). For the case at hand, namely kaon production in heavy ion collisions, isospin symmetry suggests that the production and absorption cross-section for K^+ and K^0 are roughly the same, i.e. $L_1 = L_2$ and $\epsilon_1 = \epsilon_2$. Here we have assumed that due to isospin symmetry the collision rates for both kaon species are identical.

To summarize the effect of the second species leads to a sub-leading correction $\sim \epsilon_{K^0}/6 \simeq \langle K^0 \rangle /6$ which in the present context is negligible.

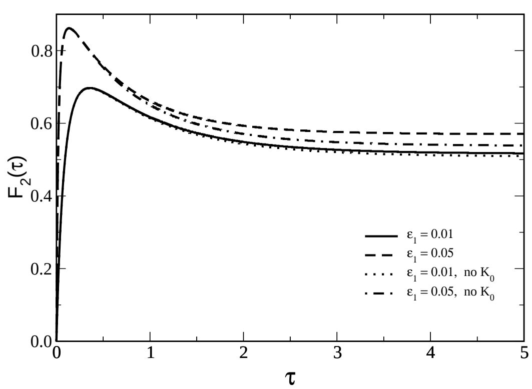

FIG. 5. Comparison of time evolution of f_2 for one and two particle species assuming binomial initial conditions and $N_0 = 0.01$

Let us now turn to the question of how to measure these fluctuations in experiment. Since we are considering rare particles, the measurement of fluctuations are obviously difficult. Here we propose to study the second factorial moment. Therefore, besides the inclusive particle number $\langle N \rangle$ one also has to measure $\langle N(N-1) \rangle$. The latter expectation value, however, is directly related to the two particle density

$$\rho_2(p_1, p_2) = \frac{d^2 N}{d^3p_1\, d^3p_2} \tag{39}$$

$$\langle N(N-1) \rangle = \int d^3p_1\, d^3p_2\, \rho_2(p_1, p_2). \tag{40}$$

It is interesting to note (see also [12]) that the same information enters the measurement of so called HBT or Bose-Einstein (BE) correlations [13]. The BE-correlation function as a function of the relative momentum is defined as

$$C_{BE}(q) = \frac{\rho_2(q)}{\rho_{11}(q)}, \tag{41}$$

where

$$\rho_2(q) \equiv \int dp_1 dp_2 \rho_2(p_1, p_2) \delta(|p_1 - p_2| - q) \tag{42}$$

and

$$\rho_{11}(q) \equiv \int dp_1 dp_2 \rho_1(p_1) \rho_2(p_2) \delta(|p_1 - p_2| - q) \tag{43}$$

Usually the C_{BE} is parameterized as

$$C_{BE}(q) = 1 + \lambda e^{-q^2 R^2} \tag{44}$$

so that outside the correlation region $q \gg 1/R_{source}$ the correlation function assumes a value of one. However, in case of rare particle subject to $U(1)$ conservation law, i.e. kaons at low energy heavy-ion collisions, this will be different:

In terms of $\rho_2(q)$ and $\rho_{11}(q)$ the factorial moment F_2 is given by

$$F_2 = \frac{\int dq \rho_2(q)}{\int dq \rho_{11}(q)} \to \frac{1}{2}, \tag{45}$$

which, as shown, assumes a value of $F_2 = 1/2$ in equilibrium as a result of strangeness conservation. Using, on the other hand, the standard parameterization for the BE-correlation function, one obtains

$$\rho_2(q) = \rho_{11}(q)(1 + \lambda e^{-q^2 R^2}) \tag{46}$$

and hence

$$F_2 = \frac{\int dq \rho_2(q)}{\int dq \rho_{11}(q)} = 1 + \lambda \int dq e^{-q^2 R^2} \rho_{11}(q) > 1, \tag{47}$$

where the second term is only a very small correction of the order of a few percent, given a typical source size of $R_{source} \simeq 5\,\mathrm{fm}$ [13]. Obviously the standard parameterization for the BE-correlation function is not adequate for the case of rare particles subject to a conservation law. Since the correlations due to strangeness conservation are not expected to introduce any momentum dependence, we thus predict, that the BE-correlation for rare particles subject to $U(1)$ conservation should asymptotically approach a value of 1/2, i.e.

$$C_{BE}(q \gg 1/R_{source}) \simeq \frac{1}{2}. \tag{48}$$

Therefore, the rather difficult measurement of kaon correlations at heavy-ion collisions at 1 AGeV would not only provide information about the size of the kaon emitting source but, more importantly, would also be a direct measurement of the degree of equilibrium reached in these collisions.

VI. CONCLUSION

In this work, we addressed the question of number fluctuations of rarely produced particles. We carried out the analysis by solving the master equation derived in Ref.

[9]. The most important aspect of the master equation (1) is that we can treat the conservation laws that govern the rare particles exactly. For instance, the strangeness conservation for kaon production at the SIS energy can be treated in this way. Comparing the results of sections III and IV certainly shows the difference such a constraint makes. In previous papers, some of us explored the consequence of requiring the exact conservation on the behavior of the average multiplicity in equilibrium as well as in evolving systems. In this work, we investigated the time evolution of the second factorial moment $F_2 = \langle N(N-1)\rangle / \langle N\rangle^2$ to explore the possibility of using it as a non-equilibrium measure.

To cover a wide range of physical phenomena, we studied two extreme cases. (i) The initial population of the rare particle is much larger than the equilibrium population. (ii) The initial population is much smaller. Our main conclusion is that the measurement of F_2 can certainly tell us if the equilibrium has not been reached. Moreover, the approach of the second factorial moment towards the equilibrium depends very much on the initial condition. Assuming that the equilibrium population ϵ is very small, we see that the smaller initial population results in the approach from above to the equilibrium value ($F_2^{\rm eq.} = 1/2 + \mathcal{O}(\epsilon)$). On the other hand a larger initial population results in the approach from below with a long period of very small F_2. Hence, the experimental value of F_2 can immediately tell us if the initial population was smaller ($F_2^{\rm exp} > 0.5$), larger ($F_2^{\rm exp} < 0.5$, possibly $F_2^{\rm exp} \ll 0.5$) or the system has already reached the equilibrium before the freeze-out ($F_2^{\rm exp} \simeq 0.5$).

In summary: the essential point of this work is that in a system in thermal equilibrium the average particle number fixes the fluctuations. In case of rare particles subject to $U(1)$ charge conservation these fluctuations are different from the simple Poisson type finite number fluctuations and thus provide a measure of the degree of equilibration reached in a system. This measurement can either be achieved by measuring the inclusive one and two particle densities or via the well known Bose-Einstein correlations.

ACKNOWLEDGMENT

We thank Z. Lin for discussions. This work was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of Nuclear Physics, the Office of Basic Energy Science, Division of Nuclear Science, of the U.S. Department of Energy under Contract No. DE-AC03-76SF00098, the Committee for Scientific Research (KBN-2P03B 03018), and NSFC under project 19928511. S.J's work was supported in part by the Natural Sciences and Engineering Council of Canada and by le Fonds pour la Formation de Chercheurs et l'Aide à la Recherche du Quèbec.

APPENDIX A: INITIAL TIME EVOLUTION FOR SMALL INITIAL PARTICLE NUMBERS

The leading time dependence of the probabilities $P_n(\tau)$ can be obtained by Taylor expansion around the initial time $\tau = 0$

$$P_n(\tau) = P_n(\tau = 0) + \sum_m \frac{1}{m!} \frac{d^m P_n}{d\tau^m}\bigg|_{\tau=0} \tau^m. \tag{A1}$$

The time derivatives can be obtained by iteratively applying Eq.(1). To order τ_α^3 one obtains

$$\begin{aligned} \langle N(N-1)\rangle &= \epsilon^2\alpha^2 \left((2\tau_\alpha + \tau_\alpha^2) - \alpha(5\tau_\alpha^2 - \frac{5}{3}\tau_\alpha^3) + \mathcal{O}(\alpha^2) \right) \\ \langle N\rangle^2 &= \epsilon^2\alpha^2 \left((1+\tau_\alpha)^2 - \alpha(2\tau_\alpha + 3\tau_\alpha^2 + \tau_\alpha^3) + \mathcal{O}(\alpha^2) \right) \end{aligned} \tag{A2}$$

where we have neglected higher orders in the small variable

$$\alpha \equiv \frac{N_0}{\epsilon} \simeq \frac{N_0}{\langle N_{\rm eq.}\rangle} \ll 1. \tag{A3}$$

$$\tag{A4}$$

We have also rescaled the time τ_α according to

$$\tau = \tau_\alpha\, \alpha. \tag{A5}$$

Initially, at $\tau = 0$, the factorial moment starts out at zero

$$F_2(\tau = 0) = \frac{\langle N(N-1)\rangle}{\langle N\rangle^2} = 0. \tag{A6}$$

However, after a very short time of the order of $\tau = \frac{N_0}{N_{\rm eq.}}$ corresponding to $\tau_\alpha = 1$ the factorial moment assumes a value

$$F_2(\tau_\alpha = 1) \simeq \frac{3}{4} \tag{A7}$$

which is larger than the final equilibrium result

$$F_2(\tau \to \infty) \simeq \frac{1}{2}. \tag{A8}$$

Therefore, one expects that the factorial moment F_2 reaches a maximum value of about $F_2 \simeq 1$ after time of the order of $\tau = \frac{N_0}{N_{\rm eq.}}$. Furthermore, F_2 approaches equilibrium from above thus a measurement of $F_2 > 1/2$ will indicate that equilibrium has not been reached in a heavy ion collision.

APPENDIX B: PERTURBATIVE SOLUTION

To solve (Eq.(15))

$$\frac{\partial g}{\partial \tau} = (1-x)\,(xg'' + g' - \epsilon g) \tag{B1}$$

perturbatively, we first make an ansatz

$$g(\tau,x) = g_{\text{eq.}}(x) + \sum_{n=1}^{\infty} e^{-n^2\tau} a_n\, h_n(x) \tag{B2}$$

with

$$h_n = h_n^{(0)} + \epsilon h_n^{(1)} + \epsilon^2 h_n^{(2)} + \cdots \tag{B3}$$

By substituting the expression (B2) into Eq.(B1) and collecting the same powers of ϵ, one can easily show that the equation for $h_n^{(0)}$ is

$$(1-x)\left(x h_n^{(0)\prime\prime} + h_n^{(0)\prime}\right) + n^2 h_n^{(0)} = 0 \tag{B4}$$

which has the solution

$$h_n^{(0)}(x) \equiv F(n,-n;1;x) = \sum_{s=0}^{n} \frac{\prod_{i=0}^{s-1}(i^2-n^2)}{(s!)^2} x^s \tag{B5}$$

where $F(n,-n;1;x)$ is a hypergeometric function. All other $h_n^{(s)}$'s are determined by the functional relations

$$-n^2 h_n^{(s)} = (1-x)\left(x h_n^{(s)\prime\prime} + h_n^{(s)\prime} - h_n^{(s-1)}\right) \tag{B6}$$

Keeping terms up to ϵ and $e^{-4\tau}$ yields the following expressions for the relevant averages

$$\begin{aligned}\langle N\rangle &= g'(\tau,1)\\ &= \langle N\rangle_{\text{eq.}} + \left(-1+\frac{\epsilon}{5}\right) a_1 e^{-\tau} + \left(2-\frac{2\,\epsilon}{65}\right) a_2 e^{-4\tau}\end{aligned} \tag{B7}$$

$$\begin{aligned}\langle N(N-1)\rangle &= g''(\tau,1)\\ &= \langle N(N-1)\rangle_{\text{eq.}} - \frac{2\,\epsilon}{5} a_1 e^{-\tau} + \left(6+\frac{2\,\epsilon}{13}\right) a_2 e^{-4\tau}\end{aligned} \tag{B8}$$

To make a statement on how the averages behave as a function of time, one needs to know the coefficients a_n. Obviously, the initial condition fixes these coefficients

$$g(0,x) = g_{\text{eq.}}(x) + \sum_{n=1}^{\infty} a_n\, h_n(x) = \sum_{s=0}^{\infty} x^s P_s(0) \tag{B9}$$

In general, to get a_n, one needs to invert

$$P_s(0) = \sum_{n=s}^{\infty} A_{sn}\, a_n + \mathcal{O}(\epsilon) \tag{B10}$$

where

$$A_{sn} = \frac{1}{(s!)^2} \prod_{i=0}^{s-1} (i^2 - n^2) \tag{B11}$$

It is not a trivial task in general to solve the above equation for all a_n. A simple procedure to solve for a_n's can be given only if we ignore the $\mathcal{O}(\epsilon)$ corrections and also if there is a last index N for which P_N is non-zero. In that case, we can write

$$P_s(0) = \sum_{n=s}^{N} A_{sn}\, a_n \tag{B12}$$

This is a triangular linear system of equations and can be easily solved by first getting $a_N = P_N(0)/A_{NN}$ and then a_{N-1} and so on.

To have a general understanding of how a_n's behave, first consider the condition

$$\begin{aligned}1 = \sum_{s=0}^{\infty} P_s(0) &= \sum_{s=0}^{\infty}\sum_{n=s}^{\infty} A_{sn}\, a_n\\ &= \sum_{n=0}^{\infty} a_n \sum_{s=0}^{n} A_{sn}\end{aligned} \tag{B13}$$

From Eqs. (B4), (B5) and (B11), it is easy to see that

$$\sum_{s=0}^{n} A_{sn} = h_n^{(0)}(1) = \delta_{0n} \tag{B14}$$

Hence, the above condition constrains

$$a_0 = 1 \tag{B15}$$

Otherwise, a_n's only have to make $0 \le P_s(0)$, or

$$0 \le \sum_{n=s}^{\infty} A_{sn}\, a_n \tag{B16}$$

If s is odd then $A_{sn} < 0$ for all $n \ge s$ and if s is even then $A_{sn} > 0$ for all $n \ge s$. To keep the probabilities positive, a_n's must have alternating signs starting with $a_1 < 0$ and $|a_n|$ must be a monotonic decreasing function of n. Furthermore, to keep the probabilities finite, $|a_n|$ must decrease faster than any power of n.

Numerical investigation shows that the size of a_s is $\mathcal{O}(1)$ up to $s \simeq \sqrt{N_0}$ and from then on $|a_s|$ falls like a Gaussian (faster for larger $N_0 \gtrsim 15$). Empirically,

$$a_n = (-1)^n 2\exp\left(-n^2/M_0\right) \qquad (n \ge 1) \tag{B17}$$

where $M_0 \approx N_0$ works up to $N_0 \simeq 15$. For larger N_0, some small P_s can become negative. With $M_0 = 15$, the above expression (B17) gives

$$\begin{aligned}&a_1 = -1.87101 \quad a_2 = 1.53186 \quad a_3 = -1.09762\\ &a_4 = 0.688308 \quad a_5 = -0.377751\end{aligned} \tag{B18}$$

These coefficients result in a probability distribution with $\langle N\rangle_{\tau=0} = 14.8$ and $\langle N(N-1)\rangle_{\tau=0} = 210.2$ as shown in Fig. 6.

Using the initial distribution given by a Poisson distribution with $N_0 = 15$ and solving Eq. (B12) yield

$$a_1 = -1.86667 \quad a_2 = 1.52 \quad a_3 = -1.08444,$$
$$a_4 = 0.682074 \quad a_5 = -0.380919 \tag{B19}$$

These of course result in $\langle N(N-1)\rangle_{\tau=0} = N_0^2 = 225$.

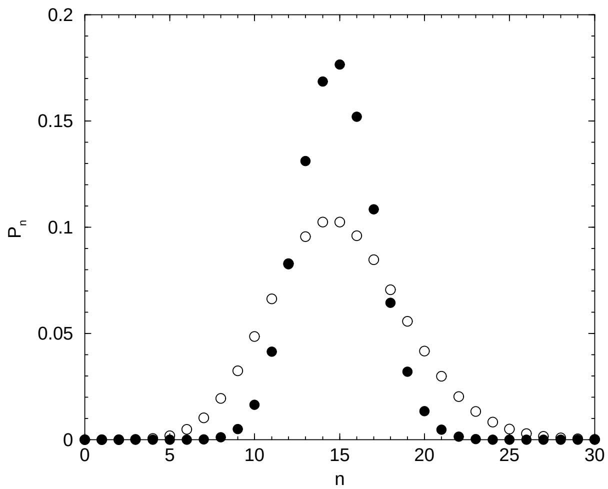

FIG. 6. The filled circles represents the probability distribution resulting from using Eq. (B17) with $M_0 = 15$. The open circles represents a Poisson distribution with $N_0 = 15$.

For completeness we also quote the results with a Poissonian initial distribution with $N_0 = 5$, solving Eq. (B12) yields

$$a_1 = -1.6027 \quad a_2 = 0.871375 \quad a_3 = -0.347491$$
$$a_4 = 0.107955 \quad a_5 = -0.0272918 \tag{B20}$$

These values are used to calculate the perturbative solutions shown in Fig. 4. The Gaussian formula gives

$$a_1 = -1.63746 \quad a_2 = 0.898658 \quad a_3 = -0.330598$$
$$a_4 = 0.0815244 \quad a_5 = -0.0134759 \tag{B21}$$

[1] R. Hagedorn, CERN yellow report 71-12, 101 (1971).
[2] E.V. Shuryak, Phys. Lett. **B42**, 357 (1972); Sov. J. Nucl. Phys. **20**, 295 (1975).
[3] U. Heinz, Nucl. Phys. **A661**, 349 (1999); R. Stock, Phys. Lett. **B456**, 277 (1999).
[4] P. Braun-Munzinger, I. Heppe and J. Stachel, Phys. Lett. **B465**, 15 (1999); P. Braun-Munzinger, J. Stachel, J. P. Wessels and N. Xu, Phys. Lett. **B344**, 43 (1995); Phys. Lett. **B365**, 1 (1996); J. Cleymans and K. Redlich, Phys. Rev. Lett. **81**, 5284 (1998); D. Yen and M.I. Gorenstein, Phys. Rev. **C59** 2788 (1999); J. Letessier and J. Rafelski, Int. J. of Mod. Phys. **E9** 107 (2000).
[5] J. S. Hamieh, K. Redlich and A. Tounsi, Phys. Lett. **B486**, 61 (2000).
[6] J. E.L. Bratkovskaya, W. Cassing, C. Greiner, M. Effenberger, U. Mosel and A. Sibirtsev, Nucl. Phys. **A675**, 661 (2000).
[7] J. Cleymans, H. Oeschler and K. Redlich, Phys. Lett. **B485** 27 (2000).
[8] W. Cassing, Nucl. Phys. **A661** 486c (1999).
[9] C.M. Ko et al., nucl-th/0010004, Phys. Rev. Lett. in print.
[10] H. Oeschler, J. Phys. **G27** , 257 (2001).
[11] Z. Lin and C.M. Ko, nucl-th/0103071
[12] M. Tannenbaum, Nucl. Phys B (Proc. Suppl.) **71** 297 (1999).
[13] see e.g. U.A. Wiedemann and U. Heinz, Phys. Rept. **319** 145 (1999); U. Heinz and B.V. Jacak, Ann. Rev. Nucl. Part. Sci. **49**, 529 (1999)
[14] K. Tsushima, S.W. Huang, and A. Faessler, J.Phys. **G21** 33 (1995).
[15] C. Greiner and S. Leupold, nucl-th/0009036
[16] G. E. Brown, M. Rho and C. Song nucl-th/0010008.
[17] P. Braun-Munzinger and J. Stachel, Phys.Lett. **B490** 196 (2000).